A Construction of 2-to-1 Optical FIFO Multiplexers by a Single Crossbar Switch and Fiber Delay Lines

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Abstract

In this paper, we propose a new construction for an optical 2-to-1 FIFO multiplexer by Switched Delay Lines (SDL). We consider an $(M+2) \times (M+2)$ crossbar switch and M fiber delay lines with delay $d_1, d_2, \ldots, d_M$. These M fiber delay lines are connected from the M outputs of the crossbar switch to the M inputs of the switch, leaving two inputs (resp. two outputs) of the switch for the two inputs (resp. two outputs) of the 2-to-1 multiplexer. We show that such a construction can be operated as a 2-to-1 FIFO multiplexer with buffer $\sum_{i=1}^{M} d_i$ if (i) the delay lines are chosen to satisfy $d_1 = 1$ and $d_i \leq d_{i+1} \leq 2d_i$, $i = 1, 2, \ldots, M-1$, and (ii) the routing of a packet is according to a specific decomposition of the packet delay, called the $\mathcal{C}$-transform in this paper.

Index Terms

optical multiplexers, switched delay lines, exact emulation, FIFO queues

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I. Introduction

To build high speed packet switches that scale with the speed of fiber optics, one needs to resolve conflicts of packets that compete for the same resources. Traditionally, such conflicts are resolved by first converting optical packets into electronic packets, storing them in electronic buffers, and then converting electronic packets back into optical packets. However, such an approach incurs tremendous overheads in both the O-E conversion and the E-O conversion. As such, there has been recent research interest in constructing optical buffers directly via optical Switches and fiber Delay Lines (SDL).

The idea of using SDL for buffering is to distribute optical packets (by optical switches) over fiber delay lines with various lengths so that packets that compete for the same resources can be resolved. Early works in this direction include the feedback system by Karol [10] and the CORD (contention resolution by delay lines) project [2], [3], [4]. Those early works mainly focused on the feasibility of SDL, not on the theoretical aspect of SDL. In particular, the feedback system in [10] was proposed for approximating a shared buffered switch. In [13], [11], a genuine SDL design, named COD (Cascaded Optical Delay-Lines), is proposed for exact emulation of 2-to-1 buffered First In First Out (FIFO) multiplexers by using 2×2 crossbar switches and fiber delay lines. Though it is easy to control the switches in COD, the cost in terms of the number of switches in COD is linear in the buffer size. In [8], a more efficient design, called Logarithm Delay-Line Switch, is proposed for 2-to-1 buffered FIFO multiplexers. The number of 2×2 switches needed for such an architecture is only $O(\log B)$, where B is the buffer size. More recently, it is shown in [6] that there is a recursive method for constructing a larger 2-to-1 buffered multiplexers by connecting smaller 2-to-1 buffered multiplexers. The construction in [8] is a direct result of the recursive expansion in [6]. For other related works in SDL, we refer to [9], [7], [15], [14] and references therein.

All the works on 2-to-1 buffered FIFO multiplexers are based on multistage construction of SDL elements. In this paper, we consider a much simpler construction that only uses a single optical switch with feedback as in [10], [12]. As in most works in the SDL literature, we assume that packets are of the same size. Also, time is slotted and synchronized so that every packet can be transmitted within a time slot. By so doing, packets can be "stored" in a fiber delay line with the propagation delay being an integer multiple of time slots. As defined in [6], a 2-to-1 buffered FIFO multiplexer is a network element with two input ports and two output ports. One output port is for packet departure and the other is for packet loss (due to buffer overflow). Simultaneous arrivals at the two input ports require at least one of them being "stored" at the multiplexer and then depart in the FIFO order. In other words, a 2-to-1 buffered FIFO multiplexer is simply a FIFO queue with a finite buffer and two inputs. Since the service policy is FIFO, the delay of a packet in a 2-to-1 buffered FIFO multiplexer is known upon its arrival.

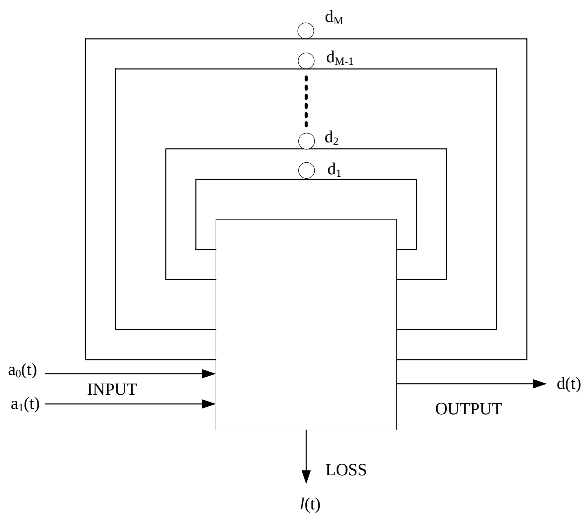

Fig. 1. A 2-to-1 multiplexer with buffer $\sum_{i=1}^{M} d_i$

To construct a 2-to-1 buffered FIFO multiplexer, in Figure 1 we consider an $(M+2)\times(M+2)$ optical crossbar switch and M fiber delay lines with delay $d_1, d_2, \ldots, d_M$. The $(M+2) \times (M+2)$ optical crossbar switch is capable of realizing all the $(M+2)!$ permutations between its inputs and outputs. The delay of these M fiber delay lines are chosen to satisfy $d_1 = 1$ and $d_i \leq d_{i+1} \leq 2d_i$, $i = 1, 2, \ldots, M-1$. Among the $M+2$ outputs of the crossbar switch, two of them are used as the output port and the loss port of the 2-to-1 multiplexer. The rest of M outputs are connected to the M fiber delay lines with delay $d_1, d_2, \ldots, d_M$, respectively. Similarly, among the $M+2$ inputs of the crossbar switch, two of them are used as the two inputs of the 2-to-1 multiplexer. The rest of M inputs are connected to the M fiber delay lines that are fed back from the outputs.

As the delay of a packet is known upon its arrival, we will use packet delay for routing. Suppose that the delay of packet that arrives at time t is x. We first find out a specific decomposition of x (called the $\mathcal{C}$-transform of x in Section II) such that

$$x = d_{i_1} + d_{i_2} + \ldots + d_{i_k}, \tag{1}$$

with $i_1 < i_2 < \ldots < i_k$. The decomposition is started from the *largest* delay line, i.e., d_M. If x is not smaller than d_M, then the M^{th} delay line is selected. We then subtract d_M from x and compare it with the second largest delay line d_{M-1}. If the remaining value is still not smaller than d_{M-1}, the second largest delay line is also selected. The process is then repeated until we have a decomposition of x as in (1). Note that the maximum delay that can be decomposed by this approach is $\sum_{i=1}^{M} d_i$. Thus, if a packet sees $\sum_{i=1}^{M} d_i$ packets in the 2-to-1 multiplexer upon its arrival, the packet is lost and it is routed to the loss port immediately.

Unlike the decomposition, the routing of a packet with delay x is started from the *smallest* delay line. Suppose that the packet delay x can be decomposed as in (1). The packet arriving at time t is routed to the delay line with delay d_{i_1} at time t, the delay line with delay d_{i_2} at time $t+d_{i_1}$, ..., and the delay line with delay d_{i_k} at time $t+\sum_{\ell=1}^{k-1} d_{i_\ell}$. By so doing, we achieve the exact delay of that packet. The only problem left is whether there is a collision (i.e., more than one packets are routed to the same fiber delay line at the same time).

The main contribution of our paper is to provide a formal proof to show that there is no collision at any fiber delay line at any time under the assumption that the M fiber delay lines are chosen to satisfy $d_1 = 1$ and $d_i \le d_{i+1} \le 2d_i$, $i = 1, 2, \ldots, M-1$. Hence, our construction is indeed an exact emulation of a 2-to-1 buffered FIFO multiplexer with buffer $\sum_{i=1}^{M} d_i$. The construction is more general than those in [8], [6] in the sense that there is no need to require $d_{i+1} = 2d_i$ for all i. We also note that the same architecture (with a different assumption on the delay lines and a different routing policy) was previously used in [12] to construct a priority queue. Our proof requires to derive a minimum distance property and a maximum distance property for packets that are routed to the same delay lines. These properties will be shown in details in Section II. With the minimum distance property and the maximum distance property, we then show that there is no collisions in Section III.

II. $\mathcal{C}$-TRANSFORM

In this section, we introduce the $\mathcal{C}$-transform that is used for obtaining the decomposition of packet delay.

Definition 1 *Consider an M-vector* $\mathcal{D}_M = (d_1, d_2, \ldots, d_{M-1}, d_M)$ *with* $d_i \in \boldsymbol{N}$*. Define a mapping* $\mathcal{C} : x \in \{0\} \cup \boldsymbol{N} \mapsto 2^M$ *as follows:*

$$\mathcal{C}(x) = \Big(I_1(x), I_2(x), \ldots, I_{M-1}(x), I_M(x)\Big), \tag{2}$$

where

$$I_M(x) = \begin{cases} 1 & \text{if } x \ge d_M \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

and for $i = M-1, \dots, 2, 1$,

$$I_i(x) = \begin{cases} 1 & \text{if } x - \sum_{k=i+1}^{M} I_k(x) \cdot d_k \geq d_i \\ 0 & \text{otherwise} \end{cases} . \tag{4}$$

We call $\mathcal{C}(x)$ the $\mathcal{C}$-transform of x (with respect to $\mathcal{D}_M$). To understand the intuition of the $\mathcal{C}$-transform, we note that the $\mathcal{C}$-transform of x is in fact a decomposition of x (like the famous Gram-Schmidt process [1]) and $\mathcal{C}(x)$ can be viewed as the coordinate vector of x with respect to $\mathcal{D}_M$. In every step of this decomposition algorithm, we compare the remaining value from the last iteration, i.e., $x - \sum_{k=i+1}^{M} I_k(x) \cdot d_k$ with the relative element in $\mathcal{D}_M$, i.e., d_i. If the remaining value is large than or equal to the relative element of $\mathcal{D}_M$, we set the corresponding element, i.e., $I_i(x)$, to 1. Otherwise, it is set to 0. For instance, if $\mathcal{D}_M = (1, 2, 2^2, \dots, 2^{M-1})$, then for all $0 \leq x \leq 2^M - 1$, $\mathcal{C}(x)$ is simply the binary representation of x (from the least significant bit to the most significant bit). To further illustrate the concept of the $\mathcal{C}$-transform, we carry out the decomposition for various packet delay in the following example.

Example 2 ($\mathcal{C}$**-transform)** Consider the 5-vector $\mathcal{D}_5 = (1, 2, 3, 6, 10)$. Then the $\mathcal{C}$-transform of x for $0 \leq x \leq 22$ is shown in Table I.

Next, we define the inverse $\mathcal{C}$-transform.

Definition 3 *The inverse mapping* $\mathcal{C}^{-1} : 2^M \mapsto \{0\} \cup \boldsymbol{N}$ *is defined as follows:*

$$\mathcal{C}^{-1}\Big(\mathcal{C}(x)\Big) = \sum_{k=1}^{M} I_k(x) \cdot d_k. \tag{5}$$

We call $\mathcal{C}^{-1}\Big(\mathcal{C}(x)\Big)$ *the inverse* $\mathcal{C}$*-transform of* $\mathcal{C}(x)$*.*

The inverse $\mathcal{C}$-transform of $\mathcal{C}(x)$ is simply the summation of the product of d_i and its coordinate $I_i(x)$. For instance, if $\mathcal{D}_M = (1, 2, 2^2, \dots, 2^{M-1})$, then for all $0 \leq x \leq 2^M - 1$, we have $\mathcal{C}^{-1}\Big(\mathcal{C}(x)\Big) = x$ as $\mathcal{C}(x)$ is the binary representation of x. However, in general, we may not have

$$\mathcal{C}^{-1}\Big(\mathcal{C}(x)\Big) = x \tag{6}$$

for any $\mathcal{D}_M$. Intuitively, one may view $\mathcal{C}^{-1}\Big(\mathcal{C}(x)\Big)$ as the "projection" of x onto $\mathcal{D}_M$. As we shall see later in Section II-B that one needs an additional condition in $\mathcal{D}_M$ to guarantee (6).

x	$I_1(x)$	$I_2(x)$	$I_3(x)$	$I_4(x)$	$I_5(x)$
0	0	0	0	0	0
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	1	0	1	0	0
5	0	1	1	0	0
6	0	0	0	1	0
7	1	0	0	1	0
8	0	1	0	1	0
9	0	0	1	1	0
10	0	0	0	0	1
11	1	0	0	0	1
12	0	1	0	0	1
13	0	0	1	0	1
14	1	0	1	0	1
15	0	1	1	0	1
16	0	0	0	1	1
17	1	0	0	1	1
18	0	1	0	1	1
19	0	0	1	1	1
20	1	0	1	1	1
21	0	1	1	1	1
22	1	1	1	1	1

TABLE I

$\mathcal{C}$-TRANSFORM OF x

A. General properties of the $\mathcal{C}$-transform

In this section, we show some properties of the $\mathcal{C}$-transform.

Lemma 4 *Suppose that $x, y \in \boldsymbol{N} \cup \{0\}$ and $1 \le i \le M$.*

(i) $0 \le \sum_{k=i}^{M} I_k(x) \cdot d_k \le \mathcal{C}^{-1}(\mathcal{C}(x)) \le x$.

(ii) *(Uniqueness)*

$$\sum_{k=i}^{M} I_k(x) \cdot d_k = \sum_{k=i}^{M} I_k(y) \cdot d_k \;\; if\ and\ only\ if \;\; I_k(x) = I_k(y),\ k = i, i+1, \ldots, M.$$

(iii) *(Monotonicity) If $0 \le x \le y$, then*

$$\sum_{k=i}^{M} I_k(x) \cdot d_k \le \sum_{k=i}^{M} I_k(y) \cdot d_k.$$

(iv) *If $x = \sum_{k=i}^{M} I_k(y) \cdot d_k$ for some y, then $x = \sum_{k=i}^{M} I_k(x) \cdot d_k$.*

As mentioned before, the $\mathcal{C}$-transform behaves as if it were a projection of x. Lemma 4(i) says that the "projection" of x is not larger than x itself. The "uniqueness" result in Lemma 4(ii) says that if two "projections" are the same, then their "coordinates" are also the same. The "monotonicity" result in Lemma 4(iii) shows that if x is not larger than y, then the "projection" of x is also not larger than the "projection" of y. Finally, it is shown in Lemma 4(iv) that if x is a "projection" of y, then the "projection" of x is itself. The proof of Lemma 4 is shown in Appendix A.

B. Unique representation of the $\mathcal{C}$-transform

In this section, we show that under an additional assumption described in (A1) below, a nonnegative integer x has an unique representation by the $\mathcal{C}$-transform of x.

(A1) Assume that $d_1 = 1$, and $d_i \leq d_{i+1} \leq 2d_i, i = 1, 2, \dots, M-1$.

To simplify our notation, we let

$$S_M = \sum_{k=1}^{M} d_k. \tag{7}$$

Lemma 5 (Complete decomposition) *Assume that (A1) holds. Then* $x = \mathcal{C}^{-1}\Big(\mathcal{C}(x)\Big) = \sum_{k=1}^{M} I_k(x) \cdot d_k$ *if and only if* $0 \leq x \leq S_M$.

Lemma 5 implies that if $0 \leq x \leq S_M$, then x can be completely decomposed and represented by $\mathcal{C}(x)$. As such, $\mathcal{C}(x)$ can be viewed as the coordinate vector of x with respect to $\mathcal{D}_M$.

Proof. For the *only if* part, we prove it by contradiction. Suppose that $x > S_M$. From (3) and (4), it follows that $I_k(x) = 1$ for all $k = 1, 2, \dots, M$. Thus, $\sum_{k=1}^{M} I_k(x) \cdot d_k = S_M < x$ and we have a contradiction.

For the *if* part, we prove it by induction on M. For $M = 1$, we have $0 \leq x \leq d_1 = 1$. Thus, x is either 0 or 1. If x = 0, from (3) we have $I_1(0) = 0$ and $x = 0 = I_1(x) \cdot d_1$. If x = 1, from (3) we have $I_1(1) = 1$ and $x = 1 = I_1(x) \cdot d_1$. Thus, the case with $M = 1$ holds trivially.

Now we assume that if $0 \leq x \leq S_M$, then $x = \sum_{k=1}^{M} I_k(x) \cdot d_k$ for some integer $M \geq 1$ as the induction hypothesis. For $M + 1$, there are two cases:

Case 1. $d_{M+1} \leq x \leq \sum_{k=1}^{M+1} d_k = S_{M+1}$:

In this case, from (3) we have $I_{M+1}(x) = 1$. Since $0 \leq x - I_{M+1}(x) \cdot d_{M+1} \leq S_{M+1} - d_{M+1} = S_M$. According to the induction hypothesis, we have

$$x - I_{M+1}(x) \cdot d_{M+1} = \sum_{k=1}^{M} I_k(x - I_{M+1}(x) \cdot d_{M+1}) \cdot d_k.$$

In view of (4), we note that $I_k(x - I_{M+1}(x) \cdot d_{M+1}) = I_k(x)$ for all $k = 1, 2, \ldots, M$. It then follows that

$$x = \sum_{k=1}^{M+1} I_k(x) \cdot d_k.$$

Case 2. $0 \le x < d_{M+1}$:

In this case, from (3) we have $I_{M+1}(x) = 0$. Since we assume that $d_1 = 1$ and $d_i \le d_{i+1} \le 2d_i$ for all i in (A1), it is easy to verify that

$$\begin{aligned} & d_{M+1} \le 2d_M = d_M + d_M \\ & \le d_M + 2d_{M-1} = d_M + d_{M-1} + d_{M-1} \\ & \le d_M + d_{M-1} + 2d_{M-2} \\ & \vdots \\ & \le \sum_{k=1}^{M} d_k + d_1 = \sum_{k=1}^{M} d_k + 1. \end{aligned} \tag{8}$$

Thus, the condition $0 \le x < d_{M+1}$ implies that $0 \le x \le \sum_{k=1}^{M} d_k = S_M$. As such, we can apply the induction hypothesis to show that

$$x = \sum_{k=1}^{M} I_k(x) \cdot d_k.$$

As $I_{M+1}(x) = 0$, we then have

$$x = \sum_{k=1}^{M+1} I_k(x) \cdot d_k.$$

■

From Lemma 4(ii) and Lemma 5, we have the following unique representation result for $\mathcal{C}$-transform.

Corollary 6 (Unique representation) *Assume that (A1) holds. For all $x, y \in \{0\} \cup \mathbf{N}, 0 \le x, y \le S_M$, we have*

$$x = y \;\; if \; and \; only \; if \;\; \mathcal{C}(x) = \mathcal{C}(y).$$

C. The concepts of i-partition points and i-clusters

In this section, we introduce the concepts of i-partition points and i-clusters under the $\mathcal{C}$-transform. Define the set

$$P_i \equiv \{x : I_k(x) = 0, k = 1, 2, \ldots, i, \; \forall x \in \{0\} \cup \mathbf{N} \}.$$

We call x an i-partition point if $x \in P_i$. For instance, P_3 in Example 2 is the set $\{0, 6, 10, 16\}$. In the following lemma, we show several properties of i-partition points.

Lemma 7

(i) The minimum i-partition point is 0, i.e., $\min_{x \in P_i}[x] = 0$.

(ii) The maximum i-partition point is $\sum_{k=i+1}^{M} d_k$, i.e., $\max_{x \in P_i}[x] = \sum_{k=i+1}^{M} d_k$.

(iii) If $x \in P_i$, then $x \in P_k$, for all $k = 1, 2, \ldots, i$.

(iv) If $x = \sum_{k=i+1}^{M} I_k(y) \cdot d_k$ for some y, then x is an i-partition point, i.e., $x \in P_i$.

Proof. Note that (i), (ii), and (iii) follows trivially from the definition of the i-partition points. Here we only prove (iv). From Lemma 4(iv), we have $x = \sum_{k=i+1}^{M} I_k(x) \cdot d_k$. In view of (4), we have $I_j(x) = 0$, for all $j = 1, 2, \ldots, i$. Thus, $x \in P_i$. ■

Define

$$N_i(x) = \min_{y \in P_i, y > x} [y] \tag{9}$$

as the i-partition point that is larger than x. If x is also an i-partition point, then $N_i(x)$ is the next i-partition point. In the following lemma, we show that the last $M - i$ coordinates of the elements between two successive i-partition points are the same.

Lemma 8 *Suppose that (A1) holds and that $x, y \in \{0\} \cup \mathbf{N}$, and $1 \leq i \leq M - 1$.*

(i) *If $x \in P_i$, then the last $M - i$ coordinated of x and $N_i(x)$ cannot be the same, i.e., there exists an ℓ (with $i + 1 \leq \ell \leq M$) such that $I_\ell(x) \neq I_\ell(N_i(x))$.*

(ii) *If $x \in P_i$ and $0 \leq x \leq y \leq N_i(x) - 1$, then the last $M - i$ coordinates of x and y are the same, i.e.,*

$$I_k(x) = I_k(y), \ k = i + 1, i + 2, \ldots, M.$$

Proof. (i) Since $x \neq N_i(x)$, we have from the unique representation property in Corollary 6 that $\mathcal{C}(x) \neq \mathcal{C}(N_i(x))$. Since x and $N_i(x)$ are both in P_i, we know that $I_\ell(x) = I_\ell(N_i(x)) = 0, \ell = 1, 2, \ldots, i$. Therefore, there must exist $i + 1 \leq \ell \leq M$ such that $I_\ell(x) \neq I_\ell(N_i(x))$.

(ii) We now prove $I_k(x) = I_k(y), \ k = i + 1, i + 2, \ldots, M$ for all $x \in P_i$ and $0 \leq x \leq y \leq N_i(x) - 1$. Since $x \leq y$, we have from the "monotonicity" result in Lemma 4(iii) that

$$\sum_{k=i+1}^{M} I_k(x) \cdot d_k \leq \sum_{k=i+1}^{M} I_k(y) \cdot d_k. \tag{10}$$

Let $u = \sum_{k=i+1}^{M} I_k(y) \cdot d_k$. As a result of Lemma 4(i), we know that $u \leq y$. Using Lemma 4(iv) yields

$$u = \sum_{k=i+1}^{M} I_k(u) \cdot d_k = \sum_{k=i+1}^{M} I_k(y) \cdot d_k. \tag{11}$$

In conjunction with (10), we have

$$\sum_{k=i+1}^{M} I_k(x) \cdot d_k \leq \sum_{k=i+1}^{M} I_k(u) \cdot d_k. \tag{12}$$

From Lemma 7(iv), we know that u is also an i-partition point, i.e., $u \in P_i$. Since x is assumed to be an i-partition point, according to the definition of $N_i(x)$ we also know that x is the largest i-partition point that is not greater than y, i.e.,

$$x = \max_{v \in P_i, v \leq y} [v].$$

Since $u \in P_i$ and $u \leq y$, we conclude that $u \leq x$. It then follows from the "monotonicity" result in Lemma 4(iii) that

$$\sum_{k=i+1}^{M} I_k(u) \cdot d_k \leq \sum_{k=i+1}^{M} I_k(x) \cdot d_k. \tag{13}$$

Finally, in view of (11), (12) and (13), we have

$$\sum_{k=i+1}^{M} I_k(u) \cdot d_k = \sum_{k=i+1}^{M} I_k(x) \cdot d_k = \sum_{k=i+1}^{M} I_k(y) \cdot d_k.$$

The proof is completed by using the "uniqueness" result in Lemma 4(ii). ■

Note from Lemma 7(ii) that the largest i-partition point is $\sum_{k=i+1}^{M} d_k$, i.e., $\max_{x \in P_i}[x] = \sum_{k=i+1}^{M} d_k$. Thus, for all $0 \leq y \leq S_M$, y is either not smaller than $\sum_{k=i+1}^{M} d_k$ or there is an i-partition point x such that $x \leq y \leq N_i(x) - 1$. In other words, the i-partition points partition the set $[0, S_M]$ into several segments. From Lemma 8, we know that the last $M - i$ coordinates of all the elements between two successive i-partition points are the same. In the following lemma, we further identify the elements in each segment that has $I_i(y) = 1$.

Lemma 9 *Suppose that (A1) holds and that* $0 \leq y \leq S_M$. *Then* $I_i(y) = 1$ *if and only if* $y \geq \sum_{k=i}^{M} d_k$, *or* $x + d_i \leq y \leq N_i(x) - 1$, *where* x *is an* i*-partition point (i.e.,* $x \in P_i$*).*

Proof. Since for all $0 \leq y \leq S_M$, y is either not smaller than $\sum_{k=i+1}^{M} d_k$ or there is an i-partition point x such that $x \leq y \leq N_i(x) - 1$. Thus, We only need to consider the following two cases:

Case 1. $\sum_{k=i+1}^{M} d_k \leq y$:

In this case, it is trivial to see that $I_j(y) = 1, j = i+1, i+2, \ldots, M$. Thus,

$$y - \sum_{k=i+1}^{M} I_k(y) \cdot d_k = y - \sum_{k=i+1}^{M} d_k \geq d_i$$

if and only if $y \geq \sum_{k=i}^{M} d_k$.

This then implies that

$$I_i(y) = 1 \quad if\ and\ only\ if \quad y \geq \sum_{k=i}^{M} d_k.$$

Case 2. $x \leq y \leq N_i(x) - 1$ for some $x \in P_i$:

From Lemma 8(ii), we know that

$$I_k(x) = I_k(y), \quad k = i+1, i+2 \ldots, M.$$

Thus, we have

$$\sum_{k=i+1}^{M} I_k(x) \cdot d_k = \sum_{k=i+1}^{M} I_k(y) \cdot d_k. \tag{14}$$

Since $x < N_i(x) < S_M$ and x is an i-partition point, we have from the complete decomposition lemma (Lemma 5) that

$$x = \sum_{k=i+1}^{M} I_k(x) \cdot d_k. \tag{15}$$

Using (14) and (15) yields

$$y - \sum_{k=i+1}^{M} I_k(y) \cdot d_k = y - x. \tag{16}$$

Thus, in this case $I_i(y) = 1$ if and only if $y - x \geq d_i$. ■

In view of Lemma 8 and Lemma 9, we know that if x is an i-partition point, then for y_1 and y_2 in $[x + d_i, N_i(x) - 1]$ one has $I_i(y_1) = I_i(y_2) = 1$ and $I_k(y_1) = I_k(y_2)$ for $k = i+1, i+2, \ldots, M$. For this, we can define the concept of an i-cluster. An i-cluster is the set of elements with $I_i(y) = 1$ and the same $I_k(y)$ for $k = i+1, i+2, \ldots, M$. From Lemma 9, it follows that i-clusters are either in the form $[x + d_i, N_i(x) - 1]$ for an i-partition point x or the set $\{y \geq \sum_{k=i}^{M} d_k\}$ (see Figure 2). For the set $\{y \geq \sum_{k=i}^{M} d_k\}$, we have $I_k(y) = 1$

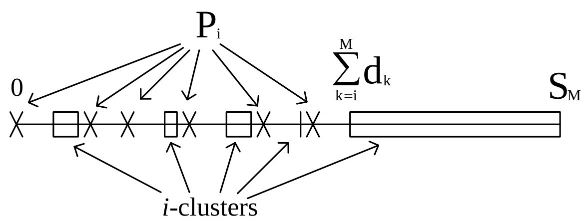

Fig. 2. An illustration graph of i-clusters

for $k = i, i+1, \ldots, M$. The set $\{y \geq \sum_{k=i}^{M} d_k\}$ is called the last i-cluster. For instance, the 3-clusters in Example 2 are the sets $\{3, 4, 5\}$, $\{9\}$, $\{13, 14, 15\}$ and $\{19, 20, 21, 22\}$.

In the following, we show two important properties of i-clusters that will be used in our proof for the 2-to-1 optical multiplexers.

Lemma 10 (i) *(Maximum distance within an i-cluster) Suppose that x and y are in the same i-cluster that is not the last cluster. Then $|x - y| \leq d_{i+1} - d_i - 1$.*

(ii) *(Minimum distance between two i-clusters) Suppose that x and y belong to two different i-clusters. Then $|x - y| \geq d_i + 1$.*

The proof of Lemma 10 is based on the following lemma on the minimum distance between two successive i-points. The proof of Lemma 11 is given in Appendix B.

Lemma 11 *Suppose that (A1) holds. For all $M \geq 2$, the difference between two successive i-partition points is not larger than d_{i+1}, i.e.*

$$if\ x \in P_i,\ then\ N_i(x) - x \leq d_{i+1}.$$

for all $1 \leq i \leq M - 1$.

Proof. (Lemma 10)

(i) Since the i-cluster being considered is not the last cluster, it must be the set $[x+d_i, N_i(x)-1]$, where x is an i-partition point. The result then follows directly from Lemma 11 that

$$|x - y| \leq N_i(x) - 1 - x - d_i \leq d_{i+1} - d_i - 1.$$

(ii) This is straightforward from Lemma 9. ∎

III. 2-TO-1 MULTIPLEXERS

In this section, we show how we construct a 2-to-1 optical buffered multiplexer. As in [6], we assume that packets are of the same size. Also, time is slotted and synchronized so that every packet can be transmitted within a time slot. The following definition was previously introduced in [6].

Definition 12 (Multiplexer) *A 2-to-1 multiplexer with buffer B is a network element with two input ports and two output ports. One output port is for departing packets and the other is for lost packets. Let $a_0(t)$ (resp. $a_1(t)$) be the number of arrival at time t from the first (resp. second) input port, $d(t)$ be the numbers of departure at time t, $\ell(t)$ be the number of loss at time t, and $q(t)$ be the number of packets queued at the multiplexer at time t (at the end of the t^{th} time slot). Then the 2-to-1 multiplexer with buffer B satisfies the following four properties:*

(P1) *flow conservation: arriving packets from the two input ports are either stored in the buffer or transmitted through the two output ports, i.e.,*

$$q(t) = q(t-1) + a_0(t) + a_1(t) - d(t) - \ell(t). \tag{17}$$

(P2) *Non-idling: there is always a departing packet if there are packets in the buffer or there are arriving packets, i.e.,*

$$d(t) = \begin{cases} 0 & \textit{if } q(t-1) = a_0(t) = a_1(t) = 0 \\ 1 & \textit{otherwise} \end{cases} . \tag{18}$$

(P3) *Maximum buffer usage: arriving packets are lost only when buffer is full, i.e.,*

$$\ell(t) = \begin{cases} 1 & \textit{if } q(t-1) = B \textit{ and } a_0(t) = a_1(t) = 1 \\ 0 & \textit{otherwise} \end{cases} . \tag{19}$$

(P4) *FIFO: packets depart in the First In First Out (FIFO) order.*

As discussed in [6], a 2-to-1 multiplexer with buffer B is simply a FIFO queue with buffer B in the queueing context. As such, it satisfies the Lindley recursion

$$q(t+1) = \min\Big[\max[q(t) + a_0(t+1) + a_1(t+1) - 1, 0], B\Big]. \tag{20}$$

Since the service policy is FIFO, one can deduce from the Lindley recursion that the delay of the first (resp. second) packet arriving at time $t+1$ is simply $q(t)$ (resp. $q(t)+1$). In other words, the delay of a packet in a 2-to-1 multiplexer is known upon its arrival.

As introduced in Section I, our construction (in Figure 1) consists of an $(M+2)\times(M+2)$ optical crossbar switch and M fiber delay lines with delay $d_1, d_2, \ldots, d_M$ that satisfies the assumption (A1), i.e., $d_1 = 1$ and $d_i \leq d_{i+1} \leq 2d_i$, $i = 1, 2, \ldots, M-1$. Among the $M+2$ outputs of the crossbar switch, two of them are used as the output port and the loss port of the 2-to-1 multiplexer. The rest of M outputs are connected to the M fiber delay lines with delay $d_1, d_2, \ldots, d_M$, respectively. Similarly, among the $M+2$ inputs of the crossbar switch, two of them are used as the two inputs of the 2-to-1 multiplexer. The rest of M inputs are connected to the M fiber delay lines that are fed back from the outputs.

As the delay of a packet is known upon its arrival, we use packet delay for routing. Suppose that the delay of a packet that arrives at time t is x. We first find out the $\mathcal{C}$-transform of x (described in the previous section). The packet will be routed to the delay line with delay d_i at $t + \sum_{k=1}^{i-1} I_k(x)\cdot d_k$ if $I_i(x) = 1$. Specifically, it will be routed to the delay line with delay d_1 at time t if $I_1(x) = 1$, and it will be routed to the delay line with delay d_2 at time $t + I_1(x)d_1$ if $I_2(x) = 1$, ... By so doing, if $x \leq S_M$, then we have from Lemma 5 that $x = \sum_{k=1}^{M} I_k(x)\cdot d_k$ and we achieve the exact delay of that packet. Note that the maximum delay for a packet is S_M and this corresponds to a FIFO queue with buffer S_M. Arrivals that see S_M packets in the 2-to-1 multiplexer are routed to the loss port. If there is no collision under such a routing policy, then our construction achieves the exact emulation of a 2-to-1 multiplexer. This is what we would like to show in the next theorem.

Theorem 13 *Suppose that our construction of the 2-to-1 multiplexer is started from an empty system. Under the routing policy, there is no collision at any fiber delay line at any time.*

Proof. Without loss of generality, packets that are lost due to buffer overflow can be excluded. It suffices to show that there is no collision in a busy period (of the FIFO queue) for the packets that are admitted to the queue. Consider two packets in the same busy period. Suppose that the k_1^{th} (resp. k_2^{th}) packet arrives at time t_1 (resp. t_2) with delay x_1 (resp. x_2). Without loss of generality, assume that $k_1 < k_2$. and $t_1 \leq t_2$. As both packets arrive in the same busy period, we have from the Lindley recursion in (20) that

$$x_2 = x_1 + (k_2 - k_1) - (t_2 - t_1). \tag{21}$$

Moreover, as there are at most two arrivals in a time slot for the 2-to-1 multiplexer, we have

$$k_2 - k_1 \leq 2(t_2 - t_1) + 1. \tag{22}$$

In conjunction with (21), it then follows that

$$x_2 - x_1 \le t_2 - t_1 + 1. \tag{23}$$

To ease our presentation, we first prove that there is no collision at i^{th} fiber delay line, for $i = 1, 2, \dots, M-1$ ($M \ge 2$) under the additional condition $d_i \ne d_{i+1}$. As a packet with delay x is routed to the i^{th} delay line only when $I_i(x) = 1$, we only need to consider packets with delay in the i-clusters. Note from the routing policy that packet k_1 (resp. k_2) will arrive at the i^{th} fiber delay line at $t_1 + \sum_{k=1}^{i-1} I_k(x_1) \cdot d_k$ (resp. $t_2 + \sum_{k=1}^{i-1} I_k(x_2) \cdot d_k$). From Lemma 5, it follows that packet k_1 (resp. k_2) will arrive at the i^{th} fiber delay line at $t_1 + x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k$ (resp. $t_2 + x_2 - \sum_{k=i}^{M} I_k(x_2) \cdot d_k$).

Case 1. x_1 and x_2 are in the same i-cluster:

Since x_1 and x_2 are in the same i-cluster, we have $I_k(x_1) = I_k(x_2)$ for all $k = i, i+1, \dots, M$. Thus, the difference of the arrival times of these two packets at the i^{th} fiber delay line is

$$\begin{aligned}
& t_2 + x_2 - \sum_{k=i}^{M} I_k(x_2) \cdot d_k - (t_1 + x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& = t_2 - t_1 + (x_2 - x_1) \\
& = k_2 - k_1 \ge 1,
\end{aligned}$$

where we use (21) in the last identity.

Case 2. x_1 and x_2 are in different i-clusters and $x_1 < x_2$:

Note that the difference of the arrival times of these two packets at the i^{th} fiber delay line is

$$\begin{aligned}
& t_2 + x_2 - \sum_{k=i}^{M} I_k(x_2) \cdot d_k - (t_1 + x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& \ge t_2 - t_1 - (x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& \ge x_2 - x_1 - 1 - (x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k),
\end{aligned} \tag{24}$$

where we use Lemma 4(i) in the first inequality and (23) in the second inequality.

Since x_1 and x_2 are in different i-clusters and $x_1 < x_2$, we have from Lemma 10(ii) that

$$x_2 - x_1 \ge d_i + 1. \tag{25}$$

Let $x_3 = \sum_{k=i}^{M} I_k(x_1) \cdot d_k$. From Lemma 4(iv), we know that $x_3 = \sum_{k=i}^{M} I_k(x_3) \cdot d_k$. As a result of Lemma 4(ii), we have $I_k(x_1) = I_k(x_3)$ for $k = i, i+1, \dots, M$. Thus, x_1 and x_3 are in the same i-cluster. Since x_1 and x_2 are in different i-clusters and $x_1 < x_2$, x_1 is not in the last i-cluster. It then follows Lemma 10(i) that

$$x_1 - x_3 \leq d_{i+1} - d_i - 1. \tag{26}$$

Using (25) and (26) in (24) yields

$$\begin{aligned}
& t_2 + x_2 - \sum_{k=i}^{M} I_k(x_2) \cdot d_k - (t_1 + x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& \geq x_2 - x_1 - 1 - (x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& \geq 2d_i - d_{i+1} + 1 \\
& \geq 1,
\end{aligned}$$

where we use the assumption $d_{i+1} \leq 2d_i$ in the last inequality.

Case 3. x_1 and x_2 are in different i-clusters and $x_2 < x_1$:

For this case, we have from (21) that the difference of the arrival times of these two packets at the i^{th} fiber delay line is

$$\begin{aligned}
& t_2 + x_2 - \sum_{k=i}^{M} I_k(x_2) \cdot d_k - (t_1 + x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& = (k_2 - k_1) + \sum_{k=i}^{M} I_k(x_1) \cdot d_k - \sum_{k=i}^{M} I_k(x_2) \cdot d_k.
\end{aligned} \tag{27}$$

Since $x_2 < x_1$, we have from the "monotonicity" result in Lemma 4(iii) that

$$\sum_{k=i}^{M} I_k(x_1) \cdot d_k \geq \sum_{k=i}^{M} I_k(x_2) \cdot d_k.$$

It then follows that

$$\begin{aligned}
& t_2 + x_2 - \sum_{k=i}^{M} I_k(x_2) \cdot d_k - (t_1 + x_1 - \sum_{k=i}^{M} I_k(x_1) \cdot d_k) \\
& \geq k_2 - k_1 \geq 1.
\end{aligned}$$

It remains to consider the case for the M^{th} fiber delay line and the case with $d_i = d_{i+1}$. For each of these two cases, there exists only an i-cluster, i.e., the last i-cluster. As such, they can be proved in a similar manner to Case 1. ■

In the following, we show by a counterexample that there is a collision if (A1) is not satisfied.

Example 14 (A counterexample for the case with $d_{j+1} > 2d_j$) In this example, we show that there is a collision at j^{th} fiber delay line if $d_1 = 1$, $d_i \leq d_{i+1} \leq 2d_i$, $i = 1, 2, \ldots, j-1$, and $d_{j+1} > 2d_j$.

As in Theorem 13, we start from an empty system, i.e., $q(0) = 0$. Consider the sample path that has two arrivals in every time slot, i.e., $a_0(t) = a_1(t) = 1$ for all $t \geq 1$. As the multiplexer is always backlogged, it is easy to see from the Lindley recursion in (20) that the delay of the first (resp. second) packet arriving at time t is $t-1$ (resp. t).

Consider two time slots t_1 and t_2 with

$$t_1 = d_{j+1} + \sum_{k=j+2}^{M} d_k \tag{28}$$

and

$$t_2 = t_1 + (d_{j+1} - 1 - \lfloor \frac{d_{j+1} - 1}{2} \rfloor). \tag{29}$$

We will show that the first packet that arrives at t_1 will collide with a packet that arrives at t_2 at the j^{th} fiber delay line.

Note that the delay of the first packet that arrives at t_1 is $t_1 - 1$. Moreover,

$$t_1 - 1 = d_{j+1} - 1 + \sum_{k=j+2}^{M} d_k \tag{30}$$

$$= (d_{j+1} - d_j - 1) + d_j + \sum_{k=j+2}^{M} d_k. \tag{31}$$

Since we assume that $d_{j+1} > 2d_j$, we have $d_{j+1} - d_j - 1 > d_j - 1 \geq 0$. According to (3) and (4), we know from (30) and (31) that $I_j(t_1 - 1) = 1$, $I_{j+1}(t_1 - 1) = 0$, and $I_k(t_1 - 1) = 1$, $k = j+2, j+3, \ldots, M$. Using these in (31) yields

$$t_1 - 1 = (d_{j+1} - d_j - 1) + \sum_{k=j}^{M} I_k(t_1 - 1) \cdot d_k. \tag{32}$$

From the routing policy and the complete decomposition property in Lemma 5, we know that this packet is routed to the j^{th} fiber delay line at time

$$\begin{aligned} & t_1 + \sum_{k=1}^{j-1} I_k(t_1 - 1) \cdot d_k \\ &= t_1 + (t_1 - 1) - \sum_{k=j}^{M} I_k(t_1 - 1) \cdot d_k \\ &= t_1 + (d_{j+1} - d_j - 1), \end{aligned} \tag{33}$$

where we use (32) in the last identity.

Suppose that d_{j+1} is an odd number. Consider the *second* packet that arrive at time t_2. In view of (28) and (29), the delay of that packet is

$$t_2 = t_1 + (d_{j+1} - 1 - \frac{d_{j+1}-1}{2})$$
$$= (\frac{d_{j+1}-1}{2} - d_j) + \sum_{k=j}^{M} d_k. \quad (34)$$

Since we assume that $d_{j+1} > 2d_j$, $\frac{d_{j+1}-1}{2} - d_j \geq 0$. We then have from (3) and (4) that $I_k(t_2) = 1$ for all $k = j, j+1, \ldots, M$. Using these in (34) yields

$$t_2 = (\frac{d_{j+1}-1}{2} - d_j) + \sum_{k=j}^{M} I_k(t_2) \cdot d_k. \quad (35)$$

Note that this packet is routed to the j^{th} fiber delay line at time

$$t_2 + \sum_{k=1}^{j-1} I_k(t_2) \cdot d_k$$
$$= t_2 + (t_2 - \sum_{k=j}^{M} I_k(t_2) \cdot d_k)$$
$$= t_2 + (\frac{d_{j+1}-1}{2} - d_j) \quad (36)$$
$$= t_1 + (d_{j+1} - d_j - 1), \quad (37)$$

where we use (35) in (36) and (29) in (37). Thus, the second packet arriving at t_2 collides with the first packet arriving at t_1 at the j^{th} fiber delay line at time $t_1 + (d_{j+1} - d_j - 1)$.

On other hand, if d_{j+1} is an even number, we then consider the *first* packet that arrive at time t_2. In view of (28) and (29), the delay of that packet is

$$t_2 - 1 = t_1 + (d_{j+1} - 1 - (\frac{d_{j+1}-1}{2} - \frac{1}{2})) - 1$$
$$= t_1 + \frac{d_{j+1}}{2} - 1 \quad (38)$$
$$= (\frac{d_{j+1}}{2} - d_j - 1) + \sum_{k=j}^{M} d_k. \quad (39)$$

Since we assume that $d_{j+1} > 2d_j$, $\frac{d_{j+1}}{2} - d_j - 1 \geq 0$. It then follows from (3) and (4) that $I_k(t_2 - 1) = 1$ for all $k = j, j+1, \ldots, M$. Using these in (39) yields

$$t_2 - 1 = (\frac{d_{j+1}}{2} - d_j - 1) + \sum_{k=j}^{M} I_k(t_2 - 1) \cdot d_k. \quad (40)$$

Note that this packet is routed to the j^{th} fiber delay line at time

$$
\begin{aligned}
& t_2 + \sum_{k=1}^{j-1} I_k(t_2 - 1) \cdot d_k \\
&= t_2 + (t_2 - 1 - \sum_{k=j}^{M} I_k(t_2 - 1)) \cdot d_k \\
&= t_2 + (\frac{d_{j+1}}{2} - d_j - 1) \qquad (41) \\
&= t_1 + (d_{j+1} - d_j - 1), \qquad (42)
\end{aligned}
$$

where we use (40) in (41) and (38) in (42). In this case, the first packet arriving at t_2 also collides with the first packet arriving at t_1 at the j^{th} fiber delay line at time $t_1 + (d_{j+1} - d_j - 1)$.

IV. Conclusions

In this paper, we proposed a construction for an optical 2-to-1 FIFO multiplexer by switched delay lines. We considered an $(M+2) \times (M+2)$ crossbar switch and M fiber delay lines with delay $d_1, d_2, \ldots, d_M$. These M fiber delay lines are connected from the M outputs of the crossbar switch to the M inputs of the switch, leaving two inputs (resp. two outputs) of the switch for the two inputs (resp. two outputs) of the 2-to-1 multiplexer. Moreover, these M delay lines are chosen to satisfy $d_1 = 1$ and $d_i \le d_{i+1} \le 2d_i$, $i = 1, 2, \ldots, M-1$. As the delay of a packet in a 2-to-1 multiplexer is known upon its arrival, the packet delay is used for routing a packet through the M fiber delay lines. For this, we proposed the $\mathcal{C}$-transform to decompose the packet delay. We showed that there are a minimum distance property and a maximum distance property for the $\mathcal{C}$-transform, and they were then be used for showing that there is no collision at any fiber delay line at any time under our routing policy. As such, we achieved an exact emulation of a 2-to-1 multiplexer with buffer $\sum_{i=1}^{M} d_i$.

Our immediate future work is to extend 2-to-1 multiplexers to N-to-1 multiplexers. For this, we will consider multistage constructions, instead of the single stage construction in this paper. Results along this line will be reported separately.

Appendix

A.

In this appendix, we prove Lemma 4.

(i) From (3), we have

$$x \ge I_M(x) \cdot d_M \ge 0. \qquad (43)$$

According to (4), we also have

$$x - \sum_{k=i+1}^{M} I_k(x) \cdot d_k \geq I_i(x) \cdot d_i.$$

Thus,

$$x \geq \sum_{k=i}^{M} I_k(x) \cdot d_k \geq 0, \; i = 1, 2, \ldots, M-1. \tag{44}$$

That (i) holds then follows from (43) and (44).

(ii) The *if* part is trivial. For the *only if* part, we will prove it by contradiction. Suppose that there exists an ℓ (with $i \leq \ell \leq M$) such that $I_\ell(x) \neq I_\ell(y)$ and $I_k(x) = I_k(y)$ for $k = \ell+1, \ldots, M$.

From the *if* part, we know that

$$\sum_{k=\ell+1}^{M} I_k(x) \cdot d_k = \sum_{k=\ell+1}^{M} I_k(y) \cdot d_k. \tag{45}$$

Here we use the convention that a summation is 0 if its lower index is larger than its upper index. Without loss of generality, assume that $x \leq y$. Thus,

$$x - \sum_{k=\ell+1}^{M} I_k(x) \cdot d_k \leq y - \sum_{k=\ell+1}^{M} I_k(y) \cdot d_k. \tag{46}$$

Since $I_\ell(x) \neq I_\ell(y)$, we know from (4) and (46) that

$$x - \sum_{k=\ell+1}^{M} I_k(x) \cdot d_k < d_\ell, \tag{47}$$

and $y - \sum_{k=\ell+1}^{M} I_k(y) \cdot d_k \geq d_\ell$. Therefore we have $I_\ell(y) = 1$. From Lemma 4(i), it follows that

$$\sum_{k=i}^{M} I_k(x) \cdot d_k \leq x = (x - \sum_{k=\ell+1}^{M} I_k(x) \cdot d_k \,) + \sum_{k=\ell+1}^{M} I_k(x) \cdot d_k. \tag{48}$$

Using (47) and (45) in (48) yields

$$\begin{aligned} &\sum_{k=i}^{M} I_k(x) \cdot d_k \\ &< d_\ell + \sum_{k=\ell+1}^{M} I_k(x) \cdot d_k \\ &= d_\ell + \sum_{k=\ell+1}^{M} I_k(y) \cdot d_k \\ &= \sum_{k=\ell}^{M} I_k(y) \cdot d_k, \end{aligned} \tag{49}$$

where we use the fact $I_\ell(y) = 1$ in the last identity. Since $I_i(x)$ and d_i are nonnegative, we have from (49) that

$$\sum_{k=i}^{M} I_k(x) \cdot d_k < \sum_{k=\ell}^{M} I_k(y) \cdot d_k \leq \sum_{k=i}^{M} I_k(y) \cdot d_k.$$

This contradicts to the assumption that $\sum_{k=i}^{M} I_k(x) \cdot d_k = \sum_{k=i}^{M} I_k(y) \cdot d_k$.

(iii) We prove this by induction on M. There are three cases for $M = 1$:

Case 1. $0 \leq x \leq y < d_1$:

In this case, we have from (3) that $I_1(x) = I_1(y) = 0$. Thus, $I_1(x) \cdot d_1 = I_1(y) \cdot d_1 = 0$.

Case 2. $0 \leq x < d_1 \leq y$:

In this case, we have from (3) that $I_1(x) = 0, I_1(y) = 1$. Thus, $I_1(x) \cdot d_1 = 0 < I_1(y) \cdot d_1 = d_1$.

Case 3. $d_1 \leq x \leq y$:

In this case, we have from (3) that $I_1(x) = I_1(y) = 1$. Thus, $I_1(x) \cdot d_1 = I_1(y) \cdot d_1 = d_1$.

Now we assume that (iii) holds for some integer $M \geq 1$. For $M + 1$, there are also three cases:

Case 1. $0 \leq x \leq y < d_{M+1}$:

In this case, we have from (3) that $I_{M+1}(x) = I_{M+1}(y) = 0$. Since $0 \leq x \leq y$, it follows from the induction hypothesis that

$$\sum_{k=i}^{M} I_k(x) \cdot d_k \leq \sum_{k=i}^{M} I_k(y) \cdot d_k.$$

Thus, we have

$$\sum_{k=i}^{M+1} I_k(x) \cdot d_k \leq \sum_{k=i}^{M+1} I_k(y) \cdot d_k.$$

Case 2. $0 \leq x < d_{M+1} \leq y$:

In this case, we have from (3) that $I_{M+1}(x) = 0, I_{M+1}(y) = 1$. From Lemma 4(i), we have

$$\begin{aligned} &\sum_{k=i}^{M+1} I_k(x) \cdot d_k \leq x \\ &< d_{M+1} = I_{M+1}(y) \cdot d_{M+1} \\ &\leq \sum_{k=i}^{M+1} I_k(y) \cdot d_k. \end{aligned}$$

Case 3. $d_{M+1} \leq x \leq y$:

In this case, we have from (3) that $I_{M+1}(x) = I_{M+1}(y) = 1$. Since

$$0 \leq x - I_{M+1}(x) \cdot d_{M+1} \leq y - I_{M+1}(y) \cdot d_{M+1},$$

we also have from the induction hypothesis that

$$\sum_{k=i}^{M} I_k(x - I_{M+1}(x) \cdot d_{M+1}) \cdot d_k \leq \sum_{k=i}^{M} I_k(y - I_{M+1}(y) \cdot d_{M+1}) \cdot d_k.$$

Note from (4) that $I_k(x - d_{M+1}) = I_k(x)$ and $I_k(y - I_{M+1}(y) \cdot d_{M+1}) = I_k(y)$ for all $k = 1, 2, \ldots, M$. Thus,

$$\sum_{k=i}^{M+1} I_k(x) \cdot d_k \leq \sum_{k=i}^{M+1} I_k(y) \cdot d_k.$$

(iv) We also prove this by induction on M. For $M = 1$, we have $x = I_1(y) \cdot d_1$. There are two cases:

Case 1. $I_1(y) = 0$:

In this case, we know $x = 0$. From (3), $I_1(x) = 0$. Thus, we have $x = 0 = I_1(x) \cdot d_1$.

Case 2. $I_1(y) = 1$:

In this case, we know $x = d_1$. From (3), $I_1(x) = 1$. Thus, we have $x = d_1 = I_1(x) \cdot d_1$.

Now we assume that (iv) holds for some integer $M \geq 1$. For $M + 1$. There are also two cases:

Case 1. $d_{M+1} \leq x$:

Note from (i) of this lemma that $x \leq y$. Thus, we have from (3) that $I_{M+1}(x) = I_{M+1}(y) = 1$. Since $x = \sum_{k=i}^{M+1} I_k(y) \cdot d_k$, it follows that $x - d_{M+1} = \sum_{k=i}^{M} I_k(y) \cdot d_k$. From the induction hypothesis, we then have

$$x - d_{M+1} = \sum_{k=i}^{M} I_k(x - d_{M+1}) \cdot d_k.$$

Note from (4) that $I_k(x - d_{M+1}) = I_k(x)$ for all $k = 1, 2, \ldots, M$. As $I_{M+1}(x) = 1$, it then follows that $x = \sum_{k=i}^{M+1} I_k(x) \cdot d_k$.

Case 2. $0 \leq x < d_{M+1}$:

In this case, we have from (3) that $I_{M+1}(x) = 0$. Since $x = \sum_{k=i}^{M+1} I_k(y) \cdot d_k < d_{M+1}$, we know that $I_{M+1}(y)$ must be 0 and thus $x = \sum_{k=i}^{M} I_k(y) \cdot d_k$. From the induction hypothesis, it then follows that

$$x = \sum_{k=i}^{M} I_k(x) \cdot d_k.$$

As $I_{M+1}(x) = 0$, we have $x = \sum_{k=i}^{M+1} I_k(x) \cdot d_k$.

B.

In this appendix, we prove Lemma 11. We prove this lemma by induction on M. For $M = 2$, there are only two cases:

Case 1. $d_1 = 1, d_2 = 1$:

In this case, $P_1 = \{0, 1\}$. Thus, $1 - 0 = 1 = d_2$.

Case 2. $d_1 = 1, d_2 = 2$:

In this case, $P_1 = \{0, 2\}$. Thus, $2 - 0 = 2 = d_2$.

From these two cases, we have shown that $N_i(x) - x \leq d_{i+1}$ for $M = 2$ and $x \in P_i$.

We assume that this lemma holds for some integer $M \geq 2$ as the induction hypothesis. Now we consider the case with M+1. First, note that $P_M = \{0, d_{M+1}\}$ and thus $d_{M+1} - 0 = d_{M+1}$. It remains to show that $N_i(x) - x \leq d_{i+1}$ for $i = 1, 2, \ldots, M-1$. To prove this via the induction hypothesis, we will establish the relationship between the set of i-partition points with respect to the M-vector $\mathcal{D}_M = (d_1, d_2, \ldots, d_M)$ and the set of i-partition points with respect to the $M+1$-vector $\mathcal{D}_{M+1} = (d_1, d_2, \ldots, d_M, d_{M+1})$. To do this, we let $P_i^M = \{x_{i,1}, x_{i,2}, \ldots, x_{i,|P_i^M|}\}$ (with $x_{i,j} < x_{i,j+1}$) be the set of i-partition points with respect to the M-vector $\mathcal{D}_M$ and $P_i^{M+1} = \{y_{i,1}, y_{i,2}, \ldots, y_{i,|P_i^{M+1}|}\}$ (with $y_{i,j} < y_{i,j+1}$) be the set of i-partition points with respect to the $M+1$-vector $\mathcal{D}_{M+1}$.

Since d_{M+1} is in P_M^{M+1}, we have from Lemma 7(iii) that d_{M+1} is also in P_i^{M+1}, for all $i = 1, 2, \ldots, M$. Without loss of generality, let us assume that $d_{M+1} = y_{i,n^*}$ for some n^*, where $1 < n^* < |P_i^{M+1}|$. We claim that P_i^{M+1} can be obtained from P_i^M as follows:

$$y_{i,k+(n^*-1)} = x_{i,k} + d_{M+1}, k = 1, 2, \ldots, |P_i^M|, \tag{50}$$

and

$$y_{i,k} = x_{i,k}, k = 1, 2, \ldots, n^* - 1. \tag{51}$$

To see (50), consider an y in P_i^{M+1}. Note that if $y \geq d_{M+1}$, then it follows from (3) that $I_{M+1}(y) = 1$. Thus, $y - I_{M+1}(y) \cdot d_{M+1} = y - d_{M+1}$. Clearly, if $y - d_{M+1}$ is in P_i^M, then y is in P_i^{M+1}. On the other hand, if $y < d_{M+1}$, then it follows from (3) that $I_{M+1}(y) = 0$. Thus,

$y - I_{M+1}(y) \cdot d_{M+1} = y$. It is trivial to note that if y is in P_i^M, then it is also in P_i^{M+1}. In Figure 3, we depict the relationship between P_i^M and P_i^{M+1}.

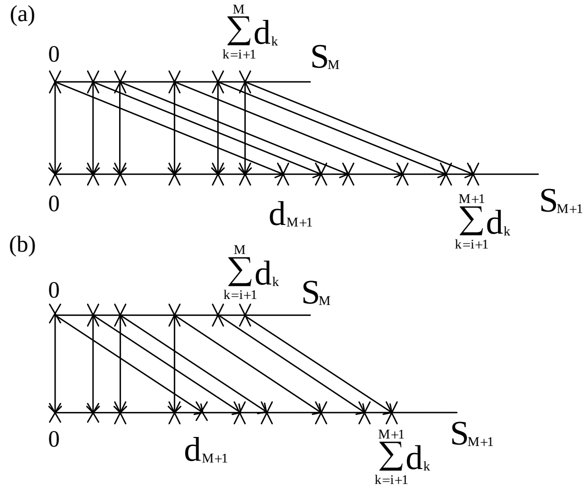

Fig. 3. The relationship of P_i^M and P_i^{M+1}: (a) $\sum_{k=i+1}^{M} d_k < d_{M+1}$ (the upper figure) and (b) $\sum_{k=i+1}^{M} d_k \geq d_{M+1}$ (the lower figure)

From the induction hypothesis and (50) and (51), we know that

$$y_{i,j} - y_{i,j-1} = x_{i,j} - x_{i,j-1} \leq d_{i+1}, \tag{52}$$

for $j \neq n^*$.

It remains to show that $y_{i,n^*} - y_{i,n^*-1} \leq d_{i+1}$. For this, we need to consider the following two cases:

Case 1. $x_{i,|P_i^M|} < y_{i,n^*}$:

In this case, we have from Lemma 7(ii) that $x_{i,|P_i^M|} = \sum_{k=i+1}^{M} d_k = y_{i,n^*-1}$. Since $d_{j+1} \leq 2d_j$ for all j, it is straightforward to see that

$$d_{M+1} \leq d_{i+1} + \sum_{k=i+1}^{M} d_k.$$

Thus,

$$y_{i,n^*} - y_{i,n^*-1} = d_{M+1} - \sum_{k=i+1}^{M} d_k \leq d_{i+1} \tag{53}$$

Case 2. $x_{i,\ell-1} < y_{i,n^*} \leq x_{i,\ell}$ for some $1 < \ell \leq |P_i^M|$:

In this case, we have $x_{i,\ell-1} = y_{i,n^*-1}$. It then follows from the induction hypothesis that

$$y_{i,n^*} - y_{i,n^*-1} \leq x_{i,\ell} - x_{i,\ell-1} \leq d_{i+1}.$$

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